THE MATHEMATICAL ARCHITECTURE OF BACH'S ''THE ART OF FUGUE''

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THE MATHEMATICAL ARCHITECTURE OF BACH'S *THE ART OF FUGUE*

Introduction

The Art of Fugue BWV 1080 is one of Johann Sebastian Bach's finest masterpieces, a true testament to his achievement as a composer. *The Art of Fugue* is the last of Bach's great monothematic cycles, after the *Musical Offering* BWV 1079, the *Goldberg Variations* BWV 988, and the *Canonic Variations on «Vom Himmel hoch da komm' ich her»* BWV 769.

Handwriting analysis and a study of watermarks in the autograph manuscript suggest that the work was probably started in the early 1740's.[1] The first known surviving autograph version was copied by the composer in 1745 and is kept at the Staatsbibliothek, Berlin, under the manuscript reference Mus. ms. Bach P200.[2] At the time of his death, Bach was supervising the publication of *The Art of Fugue*. Publication had proceeded to the point where engraver's plates had been produced but in no definitive order. Carl Philipp Emanuel Bach probably ordered the engraver's copies into what is now known as the 1751 printed sqcore. The index of both sources is summarized in Table 1 (here on p. 176). The fugues except the final one use the same subject in D minor.

In the first published version, in comparison to the earlier manuscript, two canons (BWV 1080/16-17), one complete fugue (BWV 1080/4), and one frag-

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1 See Chr. Wolff, *Zur Entstehungsgeschichte von Bachs "Kunst der Fuge"*, Ansbach, Bachwoche Ansbach, 1981, pp. 77-88; and Id., *Sulla genesi dell' "Arte della fuga"* (1983), in *Musica Poëtica. Johann Sebastian Bach e la tradizione europea*, ed. by M. T. Giannelli, Genoa, ECIG, 1986, pp. 413-427.

2 See J. S. Bach, *Die Kunst der Fuge BWV 1080: Autograph, Originaldruck*, ed. by H. G. Hoke, Leipzig, VEB Deutscher Verlag für Musik, 1979; and Id., *Die Kunst der Fuge BWV 1080*, ed. by S. Vartolo, Florence, SPES, 2008. See also Id., *Die Kunst der Fuge BWV 1080*, ed. by D. Moroney, Munich, Henle, 1989.

Table 1 – Order of pieces, shown in BWV 1080 numbers, in the Manuscript (P200, including the three appendices) and in the first printed edition (1751).

manuscript P200	1, 3, 2, 5, 9, 10*a*, 6, 7, 15 (twice), 8, 11, 14 (twice), 12,1-12,2 (aligned), 13,1-13,2 (aligned), 14 (variant), 14 (App. 1, printer's copy), 18,1 (App. 2), 18,2 (App. 2), 19 (App. 3)
first printed edition (1751)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,2, 12,1, 13,2, 13,1, 10*a*, 14, 15, 16, 17, 18,1, 18,2, 19, BWV 668*a*

mentary fugue (BWV 1080/19 included, however, in the third manuscript Appendix) were added. Also included were an early version of the tenth fugue (BWV 1080/10*a*, which corresponds to fugue VI of the autograph, whereas Counterpoint 10 BWV 1080/10 is an extension of the same piece), as well as the chorale fantasia for organ «Wenn wir in höchsten Nöten sein» (BWV 668*a*), and an arrangement for two keyboard instruments (BWV 1080/18, 1 and 2, however comprised in the second Appendix of the autograph) of an early version of one of the mirror fugues (BWV 1080/12-13). The chorale was included by the editor as a complement to the "unfinished" fourteenth fugue (BWV 1080/19), but it was not intended by Bach to be included in the work as it is in a completely different style in comparison to all the other pieces.[3] The fugues – each of which is designated as *Contrapunctus* – appear with each voice on a separate stave and most are in four parts.

One of the most intriguing problems concerning this collection is that of the ordering of its various compositions, and there has been much debate on this problem.[4] The assignment of fugue numbers in the manuscript with roman numbers is in fact posthumous.[5] The order of publication of the pieces in the 1751 edition, and in all subsequent editions, was essentially "pedagogical". Fol-

[3] See *J. S. Bach*, ed. by M. Boyd, Oxford - New York, Oxford University Press, 1999, p. 24; J. Chailley, *L'art de la fugue de J.-S. Bach: étude critique des sources, remise en ordre du plan, analyse de l'œuvre*, Paris, Leduc, 1971; H. H. Eggebrecht, *Bachs Kunst der Fuge: Erscheinung und Deutung*, München-Zürich, Piper, 1984; P. Schleuning, *Johann Sebastian Bachs "Kunst der Fuge": Ideologien, Entstehung, Analyse*, Kassel, Bärenreiter, 1993.

[4] See F. Szymichowski, *Zu den Neuordnungen von Bachs "Kunst der Fuge" durch Wolfgang Graeser und Hans Theodor David*, «Zeitschrift für Musikwissenschaft», XII, 1929/30, pp. 480-488; H. Husmann, *Die "Kunst der Fugue" als Klavierwerk. Besetzung und Anordnung*, «Bach Jahrbuch», XXXV, 1938, pp. 1-61; J. Chailley, *L'ordre des morceaux dans "L'Art de la fugue"*, «Revue de Musicologie», LIII, 1967, pp. 110-136; G. G. Butler, *Ordering Problems in J. S. Bach's "Art of Fugue" Resolved*, «Musical Quarterly», LXIX, 1983, pp. 44-61; Kl. Hofmann, *Kritischer Bericht*, in J. S. Bach, *Die Kunst der Fuge*, Kassel, Bärenreiter, 1996 («Neue Ausgabe sämtlicher Werke», Serie VIII, Band 2); E. Livingston, *Notes on the "Art of Fugue": A Fragment*, «Bach», XXVIII, 1997, pp. 82-86; G. G. Butler, *Scribes, Engravers, and Notational Styles: The Final Disposition of Bach's Art of Fugue*, in *About Bach*, ed. by G. G. Butler, G. B. Stauffer and M. D. Greer, Urbana-Chicago, University of Illinois Press, 2008, pp. 111-123.

[5] See H. G. Hoke, *Zur Handschrift Mus. ms. autogr. Bach P200 der Deutschen Staatsbibliothek Berlin*, in Bach, *Die Kunst der Fuge* cit. (see here fn. 2).

Table 2 – List of pieces included in *The Art of Fugue* (BWV 1080) along with their classification, typology, and total bars. The first column shows the identification numbers considered in this paper. The roman numbers in square brackets indicate the numeration of pieces on the autograph manuscript P200. Total bars, when not indicated, refer to the first printed edition (1751).

number	*BWV classification*	*Name*	*type*	*total bars*
1	1080/1	*Contrapunctus 1* [I]	simple fugue	78
2	1080/2	*Contrapunctus 2* [III]	simple fugue	84
3	1080/3	*Contrapunctus 3* [II]	simple fugue	72
4	1080/4	*Contrapunctus 4*	simple fugue	138
5	1080/5	*Contrapunctus 5* [IV]	counter-fugue	90
6	1080/6	*Contrapunctus 6 a 4 in stylo francese* [VII]	counter-fugue	79
7	1080/7	*Contrapunctus 7 a 4 per augment. et diminut.* [VIII]	counter-fugue	61
8	1080/8	*Contrapunctus 8 a 3* [X]	triple fugue	188
9	1080/9	*Contrapunctus 9 a 4 alla duodecima* [V]	double fugue	130
	1080/10*a*	*Contrapunctus a 4* [VI]	double fugue	100
10	1080/10	*Contrapunctus 10 a 4 alla decima*	double fugue	120
11	1080/11	*Contrapunctus 11 a 4* [XI]	triple fugue	184
12	1080/12,1 and 2	*Contrapunctus inversus 12 a 4* [XIII] and *Contrapunctus inversus a 4* [XIII]	mirror fugue	56 (*rectus*) 56 (*inversus*)
13	1080/13,1 and 2	*Contrapunctus a 3* [XIV] and *Contrapunctus inversus a 3* [XIV]	mirror and counter-fugue	71 (*rectus*) 71 (*inversus*)
–	1080/14	*Canon per augmentationem in contrario motu* [XII; XV; App. 1]	canon	109
–	1080/15	*Canon alla ottava* [IX]	canon	103
–	1080/16	*Canon alla decima contrapunto alla terza*	canon	82
–	1080/17	*Canon alla duodecima in contrapunto alla quinta*	canon	78
–	1080/18,1 and 2	*Fuga a 2 clav.* [App. 2] (*rectus*), and *Alio modo Fuga a 2 clav.* (*inversus*)	fugue for two instruments	71 (*rectus*) 71 (*inversus*)
14	1080/19	*Fuga a 3 soggetti* [App. 3]	(unfinished) triple (possibly quadruple) fugue	239 (ms) 233 (first ed.)

lowing an increasing level of musical complexity, the first block of four simple fugues was placed at the beginning, followed by three stretto fugues, four double/triple fugues, four mirror fugues, and ending with *Fuga a 3 soggetti* (BWV 1080/19), a fugue probably intended for four voices (see Table 2, here above).

Gregory Butler, in particular, has been involved in detailed research into the engraving of *The Art of Fugue*, dealing with all the cases in which tell-tale signs of erased page numbers remained in evidence on the facsimile copy.[6] He found, for example, that the augmentation canon (BWV 1080/14) would originally have been intended to follow the other three canons as the last in the group of four, and that the four canons probably followed the cycle of fourteen fugues.

Fugue BWV 1080/19 appears unfinished in manuscript P200. It cuts off peremptorily at bar 239, while in the first printed edition it cuts off at bar 233. Below bar 239, in the manuscript, there is a note written by Bach's eldest son, Carl Philipp Emanuel Bach: «Ueber dieser Fuge, wo der Nahme BACH im Contrasubject angebracht worden, ist der Verfasser gestorben». This line has generated romantic images of Bach dictating the notes of Fugue BWV 1080/19 shortly before his death, but calligraphic research has shown that this is false. Probably the editor was striving for pure sensationalism. The handwriting on the last page of Fugue BWV 1080/19 is definitely Bach's, and it is in a clear, steady hand as opposed to the erratic handwriting of Bach's final years.[7] The final page of Fugue BWV 1080/19 was definitely written several years before Bach's death.

Many researchers have found numerical symbolism in the works of Johann Sebastian Bach.[8] These studies were particularly promoted by Friedrich Smend who, in the introduction to his third book on Bach's Church Cantatas,[9] pointed out numerous examples of number symbolism. Much of this symbolism includes numbers derived from the "number alphabet" in which each letter is associated with the number of its ranking position in the alphabet. Early on in his study he drew attention to the number 14, which has since become widely known as "the Bach number", being derived from B+A+C+H = 2+1+3+8 = 14. These techniques of gematria were well known in Bach's days. Other numerical symbolisms were associated with theological numerology.

Some of these studies have been criticized for focusing on numeric symbolism, loose logic, and lack of systematic analysis – especially by Ruth Tatlow.[10]

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6 See BUTLER, *Ordering Problems* cit., and ID., *Scribes, Engravers, and Notational Styles* cit.

7 See WOLFF, *Zur Entstehungsgeschichte von Bachs "Kunst der Fuge"* cit.; ID., *Sulla genesi* cit.; ID., *Johann Sebastian Bach. La scienza della musica* (2000), Milan, Bompiani, 2003.

8 See e.g. H. NORDEN, *Proportions in Music*, «Fibonacci Quarterly», II, 1964, pp. 219-222; J.-J. DUPARCQ, *Contribution à l'étude des proportions numériques dans l'œuvre de Jean Sébastien Bach*, «Revue musicale», n. 301-302, 1977, pp. 1-59; H. A. KELLNER, *Was Bach a Mathematician?*, «English Harpsichord Magazine», II, 1978, pp. 32-36; *Nombre d'or et musique*, ed. by J.-B. Condat, Frankfurt a.M., Lang, 1988; K. VAN HOUTEN - M. KASBERGEN, *Bach et le nombre*, Liège, Mardaga, 1992.

9 See J. S. BACH, *Kirchen-Kantaten*, ed. by Fr. Smend, III: *Vom 8. Sonntag nach Trinitatis bis zum Michaelis-Fest*, Berlin-Dahlem, Christlicher Zeitschriftenverlag, 1947, pp. 5-21.

10 See R. TATLOW, *Bach and the Riddle of the Number Alphabet*, Cambridge, Cambridge University Press, 1991; ID., *The Use and Abuse of Fibonacci Numbers and the Golden Section in Musicology Today*, «Understanding Bach», I, 2006, pp. 69-85; ID. *Collections, Bars and Numbers: Analytical Coincidence or Bach's Design?*, «Understanding Bach», II, 2007, pp. 37-58.

Systematic and rigorous studies in the use of mathematical proportions in Bach's works, however, as in Tatlow, have shown that in many cases the results cannot be dismissed as arithmetical coincidence.[11] Tatlow, for example, introduced the theory of proportional parallelism in which she showed that Bach intentionally manipulated the bar structure of many of his collections so that they could relate to one another at different levels of their construction with simple ratios such as 1 : 1, 2 : 1, 1 : 2, 2 : 3.

In this essay we report a mathematical architecture of *The Art of Fugue*, based on bar counts, which shows that the whole work was conceived on the basis of the Fibonacci series and the golden ratio. A proportional parallelism is also described that shows how the same proportions were used in varying degrees of detail in the work.[12]

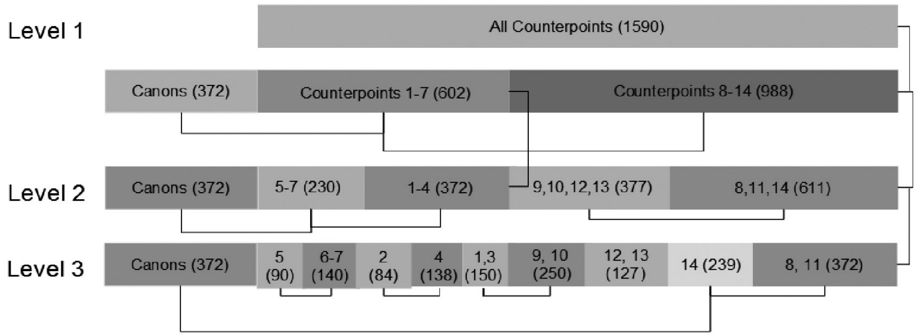

Fig. 1 – Mathematical architecture of *The Art of Fugue*.

An analysis of the bars of the pieces included in *The Art of Fugue*, reveals an architecture strongly based on the Fibonacci sequence. A phenomenon of self-similarity in the distribution of golden ratios can also be observed between more aggregate and more detailed levels of analysis. The numbers identify the Counterpoints (see Table 2). The number of bars is given in brackets. Connector lines show the golden ratios. Level 1 is the most aggregated level. In level 2 Counterpoints 1-7 and Counterpoints 8-14 are each subdivided according to the golden ratio. Level 3 shows all the golden ratios considering single or small Counterpoint aggregates. The number of bars in brackets shows a good approximation to these elements of the Fibonacci sequences: ..., 89, 144, 233, 377, 610, 987, 1597, ...

11 See Tatlow, *Collections, Bars and Numbers* cit.

12 See *ibid.*

Method and Results

In this paper we refer to Counterpoint 14 as the fugue which is classified as BWV 1080/19 (named *Fuga a 3 soggetti* in the first printed edition); Counterpoints 1-13 match BWV 1080/1-13 (see Table 2).[13]

The mathematical architecture of *The Art of Fugue* has been investigated considering the number of bars belonging to each piece, as reported in Table 2. The analysis is reported at three levels according to a top-down scheme, starting from macrostructures that aggregate the pieces at a global level (level 1), and proceeding to a more detailed level of analysis (levels 2 and 3). This three-level analysis is motivated by the observation that a fractal property of self-similarity can be found in varying degrees of detail in the work.

The total bar counts that we have considered for Counterpoints 1-13 and for the four canons are those of the printed edition (1751), here in Table 2. Counterpoint 14 is more problematic since in the manuscript the last bar (239) is incomplete, and in the printed edition it ends at the beginning of bar 233. In this paper we have considered the length of 239 bars, as in the manuscript. BWV 1080/10*a* was not considered in the analysis since it is a preliminary version of BWV 1080/10. BWV 1080/18,1 *(Fuga a 2 clav.)* and 2 *(Alio modo. Fuga a 2 clav.)* have not been included in the analysis since they are harpsichord versions of BWV1080/13,1 and BWV 1080/13,2. The chorale BWV 668*a* (*Wenn wir in hoechsten Noethen Canto Fermo in Canto*) has been excluded since it was an addendum to the first printed edition that is completely extraneous to the other material of *The Art of Fugue*.

BWV 1080/12 and BWV 1080/13 are present in two versions: *rectus* and *inversus.* The two versions are strictly related since they are obtained from the same musical material by a process of interval inversion. In the manuscript *rectus* and *inversus* for each of these two fugues were not written separately, but aligned, one under the other, as a unity. For this reason the bars for BWV 1080/12 and 13 were computed once and not as the sum of *rectus* and *inversus*.

Total bar numbers differ for some counterpoints from those of the manuscript Mus. ms. Bach P200. In particular for Counterpoints 1, 2, 3, 9, 10 the barring in the manuscript is $\frac{4}{2}$ while in the printed edition it is $\frac{2}{2}$, thus leading to a doubling of bar numbers.

The allocation of bar numbers in the printed edition is due to the fact that there is much evidence that most of the works included in the manuscript should be considered as works in progress rather than final ver-

[13] See *Thematisch-systematisches Verzeichnis der musikalischen Werke von Johann Sebastian Bach*, ed. by W. Schmieder, Wiesbaden, Breitkopf & Härtel, 1990, p. 800.

sions.[14] For example, music is written on plain paper in which scores are traced by hand with multiple drawing lines. Time signatures are sometimes inconsistent. For example Counterpoint 1 is marked alla breve ($\frac{2}{2}$) but is written in $\frac{4}{2}$. In other cases note values are diminished (as in Counterpoints 8, 11 and 12, written in $\frac{2}{4}$, and in Counterpoint 13, written in $\frac{3}{4}$).

At the beginning of Counterpoint 9 (V in the manuscript), in the Alto score, Bach has explicitly indicated the change to be made in the final version. Furthermore the first measure has been divided in $\frac{2}{2}$, as an indication of the correction to be considered throughout the work. At p. 25 of the manuscript, before the beginning of Counterpoint 8 (X in the autograph), written in half-values ($\frac{2}{4}$), it is possible to read «Folgendes muss also geschrieben werden»,[15] followed by the beginning of the same fugue with the definitive values. The preference for a diminished barring system that groups the measures 2 by 2 can be perhaps explained as means to facilitate counting the measures.

As regards fugue lengths, Counterpoints 1, 2, 3 were slightly shorter in the manuscript than in the printed edition. In Counterpoint 1 the last four measures are missing (omission of the tonic pedal and of the repetition of the theme); in Counterpoint 2 (III in the manuscript) the last six measures are missing (conclusion on the dominant A, without repetition of the theme); while in Counterpoint 3 (II in the manuscript) two measures are missing at the end.

Fibonacci Series and Φ

Mathematical analysis of *The Art of Fugue*'s structure shows a strict adherence to the Fibonacci sequence. A sequence $u_{n+2} = u_{n+1} + u_n$ (n = 1) (sequence 1) is called a generalized Fibonacci sequence.[16] If we set $u_1 = a$ and $u_2 = b$, we generate as an illustration of the sequence:

a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, ... (sequence 1)

The classical Fibonacci sequence starts with $a = 1$, $b = 1$. The first 18 terms are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ... (sequence 2)

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14 See Chailley, *L'ordre des morceaux dans "L'Art de la fugue"* cit.

15 See S. Vartolo, *Saggio introduttivo*, in Bach, *Die Kunst der Fuge BWV 1080* cit. (see here fn. 2), p. 56.

16 See R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, New Jersey, World Scientific, 1997.

Fibonacci numbers are intimately linked to the irrational number called the golden ratio:

$$\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

A generalised Fibonacci sequence has that very tidy property that all pairs of adjacent terms are in the adjacent ratio Φ but in fact it cannot be realised as any sequence of integers. Since Φ is an irrational number, it cannot be realised as a fraction of two integers, and so it is not possible to realise the ideal starting values as a pair of integers: it is only possible to approximate it.

From an historical perspective, the golden ratio was originally treated as a geometrical proportion under the name of Division in Extreme and Mean Ratio (DEMR). Ancient Greek mathematicians first studied it because of its frequent appearance in geometry, particularly in the geometry of regular pentagrams and pentagons. The Greeks usually attributed discovery of this concept to Pythagoras. The regular pentagram, which has a regular pentagon inscribed within it, was in fact the Pythagorean's symbol.

Euclid's *Elements* provide the first known written definition of what is now called the golden ratio: «A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less».[17]

Level 1 – First Subset, Second Subset, and Canons

As shown in the Method section and in Table 2, we have analyzed fourteen fugues (BWV 1080/1-13, BWV 1080/19) and four canons (BWV 1080/14-17) belonging to *The Art of Fugue*. If the corpus of fugues is divided into two groups, and we aggregate the first (1-7) and second (8-14) groups of seven fugues, and consider the aggregate total number of bars, a good approximation of the golden ratio can be found between these two subsets. The first subset (Counterpoints 1-7) encompasses a total of 602 bars, whereas the second subset encompasses a total of 988 bars:

$$\frac{\Sigma \text{ Counterpoints } 8,9,10,11,12,13,14}{\Sigma \text{ Counterpoints } 1,2,3,4,5,6,7} = \frac{988}{602} = 1.641 \approx \Phi$$

(Here Σ is taken to mean 'sum of'.)

$$\frac{\Sigma \text{ Counterpoints } 1,2,3,4,5,6,7,8,9,10,11,12,13,14}{\Sigma \text{ Counterpoints } 8,9,10,11,12,13,14} = \frac{1590}{988} = 1.609 \approx \Phi$$

17 EUCLID, *Elements*, Book 6, Proposition 30.

With regard to the Canons, the aggregate number of bars is 372, which is in perfect golden ratio to the first subset (Counterpoints 1-7):

$$\frac{\Sigma \text{ Counterpoints } 1,2,3,4,5,6,7}{\Sigma \text{ Canons}} = \frac{602}{372} = 1.618 \approx \Phi$$

The aggregated sum of the bars for the canons (372), the first subset (602), the second subset (988) and for the whole set of compositions (1590), are all numbers that closely approximate terms in the classical Fibonacci sequence 2 (Table 3).

Table 3 – Bars for Canons, Counterpoints 1-7, Counterpoints 8-14 and their closest Fibonacci numbers.

level 1	*Canons*	*Counterpoints 1-7*	*Counterpoints 8-14*
The Art of Fugue	372	602	988
Fibonacci numbers	377	610	987

A fractal phenomenon of self-similarity can be observed when analyzing the first and second subsets separately (level 2), because in each subset two subgroups of Counterpoints can be found which are in the golden ratio.

Level 2 – First Subset: Simple and Counter-fugues

The aggregate sum of bars for Counterpoints 1-7 (602) can be divided very precisely into two groups, which are in the golden ratio (Table 4, here on p. 184). The first group is composed of Counterpoints 1-4 (78+84+72+138 = 372), and the second group is composed of Counterpoints 5-7 (90+79+61 = 230):

$$\frac{\Sigma \text{ Counterpoints } 1,2,3,4}{\Sigma \text{ Counterpoints } 5,6,7} = \frac{372}{230} = 1.617 \approx \Phi$$

$$\frac{\Sigma \text{ Counterpoints } 1,2,3,4,5,6,7}{\Sigma \text{ Counterpoints } 1,2,3,4} = \frac{602}{372} = 1.618 \approx \Phi$$

The sum of Counterpoints 6 and 7 is 140 bars (79+61), whereas Counterpoint 5 is 90 bars. 90, 140, 230, 372, and 602 correspond very closely to five elements of a generalized Fibonacci sequence starting from 90, 140 (90, 140, 230, 370, 600, ...).

Table 4 – Bars for Counterpoints 1-4, Counterpoints 5-7, and their closest Fibonacci numbers.

level 2: C1-7	*C1*	*C2*	*C3*	*C4*	*C5*	*C6*	*C7*
The Art of Fugue	372				230		
Fibonacci numbers	377				233		

Level 2 – Second Subset: Double/Mirror and Triple Fugues

Counterpoints 8-14 can also be divided into two subgroups that are in the golden ratio. The aggregates of double (9-10) and mirror fugues (12-13), whose total number of bars is 377 (130+120+56+71), and the triple fugues (8, 11, 14) whose total is 611 (188+184+239) are very close to the golden ratio (Table 5).

$$\frac{\Sigma \text{ Counterpoints } 8, 11, 14}{\Sigma \text{ Counterpoints } 9, 10, 12, 13} = \frac{611}{377} = 1.620 \approx \Phi$$

$$\frac{\Sigma \text{ Counterpoints } 8, 9, 10, 11, 12, 13, 14}{\Sigma \text{ Counterpoints } 8, 11, 14} = \frac{988}{611} = 1.617 \approx \Phi$$

The aggregated sum of all fugues in the second subset in level 2 is 988. The values of 377, 611 and 988 are very close to the Fibonacci numbers 377, 610, and 987 (Table 5).

Table 5 – Bars for Counterpoints 9-13, Counterpoints 8-14 and their closest Fibonacci numbers.

level 2: C9-14	*C9*	*C10*	*C12*	*C13*	*C8*	*C11*	*C14*
The Art of Fugue	377				611		
Fibonacci numbers	377				610		

In level 3 golden ratios between single fugues and pairs of fugues are examined. The approximation to the golden ratio is somehow lower in level 3 than in the previous levels. This is possibly due to the fact that the fractional terms of the ratios are lower, being less aggregated than in the previous levels. As Φ is an irrational number, the lower the fractional terms of a ratio, the lower the approximation to the number. Furthermore, it can be suggested that at this final level the mathematical requirements had to be conciliated with more stringent musical requirements.

Level 3 – Counter-fugues 5-6-7

In counter-fugues a variation of the main subject is used in both regular and inverted form. Counterpoints 5, 6, and 7 are composed of 90, 79, and 61 bars respectively. The total number of bars for counter-fugues is 230. The total for Counterpoints 6 and 7 is 140 (79+61). All these correspond to terms of the generalized Fibonacci sequence starting with 90, 140 (90, 140, 230).

The number of bars of Counterpoint 5 (90) is in approximately golden ratio with the total of Counterpoints 6 and 7 (140) (Table 6):

$$\frac{\Sigma\text{ Counterpoints }6,7}{\text{Counterpoint }5}=\frac{140}{90}=1.555\approx\Phi$$

Furthermore, the relationship between the total of Counterpoint 6 and 7 is in approximately golden ratio with all counter-fugues:

$$\frac{\Sigma\text{ Counterpoints }5,6,7}{\Sigma\text{ Counterpoints }6,7}=\frac{230}{140}=1.642\approx\Phi$$

Table 6 – Comparison between the bars in Counterpoint 5, Counterpoints 6-7, and their closest Fibonacci numbers.

level 3: counter-fugues	*C5*	*C6*	*C7*
The Art of Fugue	90	140	
Fibonacci numbers	89	144	

Level 3 – Triple Fugues 8, 11, 14

The total number of bars for Counterpoint 14 (239) is in approximately golden ratio to the other triple fugues, 8 and 11 (372 bars):

$$\frac{\Sigma\text{ Counterpoints }8,11}{\text{Counterpoint }14}=\frac{372}{239}=1.556\approx\Phi$$

The total number of bars for Counterpoints 8, 11, 14 (611) is in approximately golden ratio to that of Counterpoints 8 and 11:

$$\frac{\Sigma\text{ Counterpoints }8,11,14}{\Sigma\text{ Counterpoints }8,11}=\frac{611}{372}=1.642\approx\Phi$$

The numbers 611 and 372 are very close to the two elements 610 and 377 of the classical Fibonacci series.

Table 7 – Bars in Counterpoint 14, Counterpoints 8 and 11, and their closest Fibonacci numbers.

level 3: triple fugues	*C14*	*C8*	*C11*
The Art of Fugue	239	372	
Fibonacci numbers	233	377	

Level 3 – Simple Fugues 2, 4

The simple fugues 2 and 4 total 84 and 138 bars respectively, which approximate the golden ratio:

$$\frac{\text{Counterpoint 4}}{\text{Counterpoint 2}} = \frac{138}{84} = 1.642 \approx \Phi$$

Table 8 – Bars in Counterpoint 2, Counterpoint 4, and their closest Fibonacci numbers.

level 3: simple fugues 2 and 4	*C2*	*C4*
The Art of Fugue	84	138
Fibonacci numbers	89	144

Symmetrical Relationships between the First and Second Subset

The ratio:

$$\frac{\Sigma \text{ Counterpoints } 6, 7}{\text{Counterpoint 5}} = \frac{140}{90} = 1.555$$

that includes counterpoints in the first subset is nearly equal to the ratio:

$$\frac{\Sigma \text{ Counterpoints } 8, 11}{\text{Counterpoint 14}} = \frac{372}{239} = 1.556$$

It can therefore be suggested that these subsets of triple counterpoints have a symmetrical function in the first and second subsets of seven counterpoints.

Similarly, the ratio:

$$\frac{\Sigma \text{ Counterpoints } 5, 6, 7}{\Sigma \text{ Counterpoints } 6, 7} = \frac{230}{140} = 1.642$$

is equal to the ratio:

$$\frac{\Sigma\ \text{Counterpoints}\ 8, 11, 14}{\Sigma\ \text{Counterpoints}\ 8, 11} = \frac{611}{372} = 1.642$$

These symmetrical relations between the first and second subset are summarized in Table 9.

Table 9 – Symmetrical relations between the first (Counterpoints 1-7) and second (Counterpoints 8-14) subset of *The Art of Fugue.* The numbers in the top row are those of Counterpoints.

1-4	*1-4*	*6/7*	*5*	*6/7*	*1-4*	*1-4*	*12/13*	*12/13*	*8/11*	*14*	*8/11*	*9/10*	*9/10*
2 simple fugues		3 counter-fugues			2 simple fugues		2 mirror fugues		3 triple fugues			2 double fugues	

The symmetrical equivalence between the first and second subset is also met considering these ratios:

$$\frac{\Sigma\ \text{Counterpoints}\ 1, 2, 3, 4}{\Sigma\ \text{Counterpoints}\ 5, 6, 7} = \frac{372}{230} = 1.617$$

is nearly equal to the ratio:

$$\frac{\Sigma\ \text{Counterpoints}\ 8, 11, 14}{\Sigma\ \text{Counterpoints}\ 9, 10, 12, 13} = \frac{611}{377} = 1.620$$

Furthermore the ratio:

$$\frac{\Sigma\ \text{Counterpoints}\ 1, 2, 3, 4, 5, 6, 7}{\Sigma\ \text{Counterpoints}\ 1, 2, 3, 4} = \frac{602}{372} = 1.618$$

is nearly equal to the ratio:

$$\frac{\Sigma\ \text{Counterpoints}\ 8, 9, 10, 11, 12, 13, 14}{\Sigma\ \text{Counterpoints}\ 8, 11, 14} = \frac{988}{611} = 1.617$$

These ratios are summarized in Table 10 (here on p. 188), which highlights the symmetrical relations between the first and second subsets.

Table 10 – Symmetrical match between first and second subset. The ratios for Counterpoints marked in light grey and dark grey are similar in both subsets.

1-4	*1-4*	*6/7*	*5*	*6/7*	*1-4*	*1-4*	*12/13*	*12/13*	*8/11*	*14*	*8/11*	*9/10*	*9/10*
2 simple fugues		3 counter-fugues			2 simple fugues		2 mirror fugues		3 triple fugues			2 double fugues	

A further symmetry between the first and second subset is determined by Counterpoints 1, 3 and Counterpoints 9, 10.

The ratio:

$$\frac{\text{Counterpoint 1}}{\text{Counterpoint 3}} = \frac{78}{72} = 1.083$$

is perfectly equal to the ratio:

$$\frac{\text{Counterpoint 9}}{\text{Counterpoint 10}} = \frac{130}{120} = 1.083$$

This symmetrical relationship is illustrated in Table 11.

Table 11 – The ratio between Counterpoints 1 and 3 is equal to the ratio between Counterpoints 10 and 9, showing a further symmetrical match between the two subsets.

1	*3*	*7*	*5*	*6*	*2*	*4*	*12*	*13*	*8*	*14*	*11*	*10*	*9*
2 simple fugues		3 counter-fugues			2 simple fugues		2 mirror fugues		3 triple fugues			2 double fugues	

Pair Matching within the Canons

The group of four canons can be divided into two pairs that share very nearly the same ratio. The ratio:

$$\frac{\textit{Canon alla ottava}}{\textit{Canon alla duodecima}} = \frac{103}{78} = 1.320$$

is nearly equal to the ratio:

$$\frac{\textit{Canon per augmentationem in contrario motu}}{\textit{Canon alla decima}} = \frac{109}{82} = 1.329$$

Discussion

The results show that the whole architecture of *The Art of Fugue* is based on the use of the Fibonacci numbers and the golden ratios, at least on the level of bar totals and grouping of the pieces. Since this mathematical architecture encompasses the whole work, with numerous occurrences, the possibility that they would have arisen *ipso facto* as a consequence of aesthetic choices or mathematical coincidences can be excluded. The significance of the mathematical architecture can probably be explained by considering the role of the work as a membership contribution to the Korrespondierende Sozietät der musicalischen Wissenschaften,[18] and to the "scientific" meaning that Bach attributed to counterpoint.[19]

In Bach's biography there is considerable evidence of a growing interest in Pythagorean philosophy.[20] Bach had been acquainted with Johann Matthias Gesner in Weimar, and in 1730 Gesner moved as Rector to the Thomasschule, where Bach was Kantor. Gesner taught Greek philosophy with an emphasis on Pythagorean thought. He even changed one of the school statutes to reflect the Pythagorean practice of repeating all one had learned during the day before retiring to bed, and the *Summa pythagorica* by Iamblichus was included by Gesner in the Thomasschule norms. It is interesting that three volumes of the *Summa pythagorica* (III-V) were devoted to arithmetic: *De communi mathematica scientia liber* (Common mathematical science), *In Nicomachi Arithmeticam introductionem liber* (Introduction to Nicomachus arithmetic), *Theologoumena arithmeticae* (Theological principles of arithmetic).[21] In all three books music is extensively treated. Bach's emphasis on numerology and numeric symbolism could easily have been derived from the perspective outlined in these books in which music is described in terms of mathematical ratios and relationships that can be found in many other domains. The study of the mathematical properties of music is understood in theological terms as a way of obtaining knowledge of the divine which is embedded in a cosmological system.

Gesner was a close friend of Bach and, since their rooms were near to each other at the school, and music played an important part in Gesner's life, we can

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[18] See H.-E. Dentler, *L' "Arte della fuga" di Johann Sebastian Bach. Un'opera pitagorica e la sua realizzazione*, Rome-Milan, Accademia nazionale di S. Cecilia - Skira, 2000; and Id., *ohann Sebastian Bachs "Kunst der Fuge". Ein pythagoreisches Werk und seine Verwirklichung*, Mainz, Schott, 2003.

[19] See D. Yearsley, *Bach and the Meanings of Counterpoint*, Cambridge, Cambridge University Press, 2002, as well as G. B. Stauffer's review of it («Journal of the AMS», LVIII, 2005, pp. 710-717).

[20] See Dentler, *L' "Arte della fuga" di Johann Sebastian Bach* cit., pp. 18-53.

[21] See Giamblico, *Summa pitagorica*, ed. by F. Romano, Milan, Bompiani, 2006 (pp. 485 ff., *passim*).

assume they had many discussions on music. Bach later dedicated his *Canon a 2 perpetuus* BWV 1075 to him. Gesner also introduced one of his pupils to Bach, Lorenz Christoph Mizler, who became one of Bach's students.

Mizler went on to create the Sozietät der musicalischen Wissenschaften and listed Bach as having joined the Society in June of 1747 (other members included Handel and Telemann). The Society devoted itself to the study of Pythagorean philosophy and the union of music, philosophy, mathematics and science. Each member had an oil portrait painted and was expected to contribute a theoretical or practical piece with the aim of developing music along the philosophical lines of Pythagoras. It is to the entry requirements of this Society that we owe both the famous 1746-48 Haussmann portrait of Bach and his above mentioned *Canonic Variations on «Vom Himmel hoch da komm' ich her»* BWV 769.

In the first publication of the Society, the *Musikalische Bibliothek*, Mizler lists Marcus Meibom's *Antiquae musicae auctores septem, graece et latine*. This book, published in Amsterdam in 1652, attempted to reconstruct the music of ancient Greece. Very probably it was known to Bach. Mizler also refers to the collected works of John Wallis, *Opera mathematica* (1693-99). Wallis was Savilian professor of geometry at Oxford University. He made many important contributions to mathematics, and Newton admitted that his development of the calculus owed much to Wallis. But in addition to his work on infinitesimals, conic sections and exponents Wallis also made specific Pythagorean references to music and harmony. References in the *Musikalische Bibliothek* were also made to Leibniz, Kepler and Robert Fludd's *Monochordum mundi symphoniacum* (1622). From his writings it is also clear that Mizler was a close friend of Bach and that they shared common interests.

The Art of Fugue was one of three pieces written for the Society, together with the *Canonic Variations* BWV 769 (a set of five canons on a popular Christmas hymn), and *The Musical Offering* BWV 1079. Bach probably began *The Art of Fugue* in 1738, and started to rework and add to it shortly before his death, or his blindness. This mingling of music and science in Bach's work has been confirmed by Wolff, who underlines that Bach the composer considered himself a musical scholar producing works of musical science at a time when Newtonian thought was very influential, especially in Leipzig.[22] Bach's music was affected by the spreading culture of Newtonianism and the general spirit of discovery that prevailed following the Scientific Revolution.

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[22] See CHR. WOLFF, *Bach's Music and Newtonian Science: A Composer in Search of the Foundations of His Art*, «Understanding Bach», II, 2007, pp. 95-106.

Tatlow has criticised the use of Fibonacci numbers and the golden section in musicology. Basing her argument on the history of mathematics, she shows that, it was theoretically possible for a composer in Bach's time to resort to the additive arithmetical sequence of Fibonacci numbers. However, this was unlikely given that the golden section and the Fibonacci sequence receive very limited coverage in the two most authoritative sources of the "state of the art" from Leipzig at Bach's time.[23] These sources were Johann Heinrich Zedler's *Großes vollständiges Universal Lexicon aller Wissenschaften und Künste*, published between 1732 and 1754, and the *Musicalisches Lexicon* by Johann Gottfried Walther (1732). The lack of coverage in these two sources led Tatlow to conclude that «there was no interest in the phenomenon, or that the new mathematical discoveries had not reached Leipzig. DEMR was probably not as divine a proportion to composers in Bach's time as some musicologists would have us think».[24]

We think that Tatlow's conclusion is right and logical in a general musical perspective, considering the majority of composers, but cannot be applied to the specific case of *The Art of Fugue* by Bach that we have described in this essay. Certainly, in the case of almost all composers, the use of Fibonacci numbers was not a common and widespread practice in composition that might be included in a universal encyclopaedia or a musical lexicon. Nor was it widespread in Bach's works, since *The Art of Fugue* is the only work in which Fibonacci numbers were used so extensively.

Tatlow herself and Roger Herz-Fischler clearly show that in Bach's time both the golden section as a numeric expression and Fibonacci numbers were known in scientific circles.[25] Bach was interested in tuning systems and organ building, and he would certainly have read Mersenne's *Harmonie universelle* (1636), which gives clear instructions on the use of the geometrical proportion associated with the golden ratio in discussing tuning systems and instrument making.

The first known calculation of the golden ratio as a decimal was given in a letter written in 1597 by Michael Maestlin, at the University of Tübingen, to his former student Johannes Kepler. He gives «about 0.6180340» for the length of the longer segment of a line of length 1 divided into the golden ratio.[26] Leonard Curchin and Herz-Fischler have also found an undated handwritten anno-

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23 TATLOW, *The Use and Abuse of Fibonacci Numbers* cit., p. 81.

24 *Ibid.*, p. 83.

25 See *ibid.*, and R. HERZ-FISCHLER, *A Mathematical History of the Golden Number*, Mineola, NY, Dover, 1998.

26 See L. CURCHIN - R. HERZ-FISCHLER, *De quand date le premier rapprochement entre la suite de Fibonacci et la division en extrême et moyenne raison?*, «Centaurus», XXVIII, 1985, pp. 129-138: 134.

tation in a sixteenth-century hand in Luca Pacioli's 1509 edition of Euclid's *Elements* (*De divina proportione*) that connects the "divine" proportion to the Fibonacci series.

Peter Schreiber has also found that Simon Jacob, who died in 1564, had published a numerical solution for DEMR.[27] The French mathematician Albert Girard also discovered the numerical solution independently; it appears in a treatise published in 1634, two years after Girard's death. Furthermore, some works by Kepler were published in Leipzig by Michael Gottlieb Hansch in 1718. In a private letter to Joachim Tanckius dated 12 May 1608, Kepler explained how he found the numerical expression of Φ from the Fibonacci sequence. In 1611 he published his discovery in *De nive sexangula* (On the six-cornered snowflake). In this essay, when speaking about dodecahedron and icosahedron, he reports that

> both of these solids, and indeed the structure of the pentagon itself, cannot be formed without this proportion that the geometers of today call divine ... It is impossible to provide a perfect example in round numbers. However, the further we advance from the number one, the more perfect the example becomes. Let the smallest numbers be 1 and 1 ... Add them, and the sum will be 2; add to this the greater of the 1s, result 3; add 2 to this, and get 5; add 3, get 8; 5 to 8, 13; 8 to 13, 21. As 5 is to 8, so 8 is to 13 approximately, and as 8 to 13, so 13 is to 21, approximately.[28]

It is clear, therefore, that the attribution of "divine" qualities to the golden section, a connotation first introduced by Pacioli was widespread in the scientific community of Kepler's epoch.

Perhaps the best proof that Bach may have known about the Fibonacci series and the golden section is Johannes Kepler's *Harmonices mundi* (1619). In this book Kepler reports his discoveries on physical harmonies in planetary motion, while other chapters are dedicated to the origin of harmonic proportions in music, to the harmonic configurations in cosmology, and to the (musical) harmony in relation to the motion of the planets. He interprets the universe in terms of harmonic proportions that can be found in music. In this treatise multiple explicit references to the "divine" proportion can be found also in association with the Fibonacci series.

For example, when discussing in Book III, Chapter XV about "Which modes or tones serve which emotions", Kepler writes:

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27 See P. SCHREIBER, *A Supplement to J. Shallit's Paper "Origins of the Analysis of the Euclidean Algorithm"*, «Historia Mathematica», XXII, 1995, pp. 422-424.

28 Cit. in HERZ-FISCHLER, *A Mathematical History of the Golden Number* cit., p. 161.

> First, then, you will remember that the hard third arose from the pentagon and the pentagon uses the division in extreme and mean ratio, which forms the divine proportion. However, the splendid idea of generation is in this proportion. For just as a father begets a son, and his son another, each like himself, so also in that division, when the larger part is added to the whole, the proportion is continued: the combined sum takes the place of the whole, and what was previously the whole takes the place of the larger part. Although this ratio cannot be expressed in numbers, yet some series of numbers may be found which continually approaches nearer to the truth; and in that series the difference of the numbers from the genuine terms (which are not countable but inexpressible) by a wonderful coincidence breeds males and females, distinguishable by the members which indicate sex. Thus if the larger part is in the first place 2, and the smaller 1, the whole is 3. Here plainly 1 is not 2 as 2 is to 3; for the difference is unity, by which the rectangle of the extremes 1 and 3 is less than equal to the square of the mean, 2. Then by adding 2 to 3 the new total becomes 5; and by adding 3 to 5 the total becomes 8, etc. The rectangle of 1 and 3 creates a female, for it falls short of the square of 2 by unity; the rectangle of 2 and 5 a male, for it exceeds the square of 3 by unity; the rectangle of 3 and 8 a female, for it falls short of the square of 5 by unity. Again from 5 and 13 arises a male, in respect of the square of 8; from 8 and 21 a female, in respect of the square of 13; and so on infinitely.[29]

Further on in Book V, Chapter II ("On the relationship of the harmonic proportion to the five regular figures"), Kepler writes again about Fibonacci numbers and their relation to the "divine" proportion:

> The second degree of relationship, which is based on origin, should be conceived as follows. First, there are some harmonic proportions of numbers which have affinity with one marriage or family, that is to say the individual perfect proportions with the cubic family. On the other hand there is a proportion which is never expressed by whole numbers, and is only demonstrated in numbers by a long series of them which gradually approach it. This proportion is called "divine", insofar as it is perfect; and it reigns in different ways through the dodecahedric marriage. Hence the following harmonies begin to sketch out this proportion: 1 : 2 and 2 : 3 and 3 : 5 and 5 : 8. For it is most imperfectly in 1 : 2, and most perfectly in 5 : 8, and would be more perfect if onto 5 and 8 added together, making 13, we were to superimpose 8, if that were not already ceasing to be harmonic.[30]

Bach could have known about these mathematical properties since *Harmonices mundi* is all focused on music theory and could certainly have caught his attention. It should also be considered that Mizler, the founder of the Korre-

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[29] J. KEPLER, *The Harmony of the World* (1619), Engl. transl. by E. J. Aiton, A. M. Duncan and J. V. Field, Philadelphia, Pa., American Philosophical Society, 1997, p. 241.

[30] *Ibid.*, p. 400.

spondierende Sozietät der musicalischen Wissenschaften was a cultivated and polymath member of the scientific community having studied theology, music composition, law, medicine and being appointed as librarian and court mathematician and member of the Erfurt Academy of Sciences.

A further possible link between Bach and Kepler could be Andreas Werckmeister, a theorist whom Bach read, admired, and paraphrased.[31] In common with Kepler and Bach, Werckmeister shared a transcendent view of counterpoint, and tied double counterpoint and canon with the orderly movement of the planets. Moreover, he specifically connected invertible counterpoint with cosmological order.[32] Analysing two of his works, we have discovered two explicit citations of Kepler's work *Harmonices mundi*: in *Musicae mathematicae hodegus curiosus* (1687) and in *Hypomnemata musica, oder Musicalisches Memorial* (1697).[33] The latter was also revised in the *Musikalische Bibliothek* directed by Mizler.[34] Access to Kepler's *Harmonices mundi* and its mathematical content could therefore have been facilitated by many sources.

The key role played by Fibonacci numbers and the golden section in the architecture of *The Art of Fugue*, along with the precision of the ratios, leave no room for doubt that these relationships emerged *per se* or as a consequence of mathematical coincidences. Therefore we are led to conclude that Bach has intentionally adopted the Fibonacci numbers in *The Art of Fugue*. Fibonacci numbers are probably used in this context as an enigma, a mathematical puzzle in which used to convey some particular "secret" meaning. These enigmas were typical of Pythagorean philosophy, reflecting its view of knowledge as a search for concealed mathematical properties.[35] (Another famous Bach enigma, linked to Bach's admission to the Sozietät der musicalischen Wissenschaften is the triple enigmatic canon for six voices in the famous above-mentioned Haussmann portrait. The canon is facing the observer but shows only three voices and the other three are missing. According to Friedrich Smend, 120 combinatorial solutions can be found for completing the canon.[36])

Bach's use of the Fibonacci numbers in the mathematical architecture of *The Art of Fugue* makes it a unique work that can only be understood by con-

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31 See YEARSLEY, *Bach and the Meanings of Counterpoint* cit., pp. 18-20.

32 See *ibid.*, p. 57.

33 A. WERCKMEISTER, *Hypomnemata musica und andere Schriften*, repr. Hildesheim - New York, Olms, 1970, respectively on p. 106 and p. 38 f.

34 See TH. PH. CALVISIUS, review of *"Hypomnemata musica, oder Musicalisches Memorial" ... von Andrea Werckmeister* (1697), in L. CH. MIZLER, *Neu eröffnete musikalische Bibliothek*, I, III, Leipzig, n.n., 1737, pp. 52-59.

35 See DENTLER, *L'"Arte della fuga" di Johann Sebastian Bach* cit., p. 54.

36 See FR. SMEND, *Johann Sebastian Bach bei seinem Namen gerufen*, Kassel, Bärenreiter, 1950, p. 21.

sidering its aim in connection with the Sozietät der musicalischen Wissenschaften. The exact numeric proportions that we have found in *The Art of Fugue* are not perceptible to listeners since they are to be found in macrostructures and not at the short-term level of auditory perception. Therefore it is more probable that there are ideological-philosophical reasons for its mathematical architecture. Future research should test the use of mathematical relationships in the internal structure of the single pieces, by investigating *The Art of Fugue* at a more detailed level of analysis. Further study is required in order to ascertain if the mathematical properties highlighted in this study are also mirrored by musical properties and if the mathematical architecture can shed further light on the intended order of pieces in *The Art of Fugue*.

RIASSUNTO – Tra le ultimissime opere di Johann Sebastian Bach, *L'arte della fuga* (BWV 1080) è considerata un capolavoro della musica d'arte occidentale. Suprema esibizione dell'ingegno contrappuntistico del compositore, essa si compone di 14 fughe e 4 canoni, di cui tuttavia Bach non ha stabilito la sequenza: nell'edizione postuma (1751) i pezzi sono stati disposti secondo una logica didattica, in ordine di difficoltà crescente. Se però esaminiamo i rapporti tra l'estensione dei pezzi, espressa in numero di battute, osserviamo che l'intero ciclo risponde a un'architettura di tipo matematico fondata sulla serie numerica di Fibonacci; osserviamo inoltre un parallelismo proporzionale dei rapporti matematici fra differenti livelli di analisi. Tale grandiosa concezione architettonica si lascia forse ricondurre al fatto che nei suoi ultimi anni Bach aderì alla Sozietät der musicalischen Wissenschaften, un sodalizio che coltivava la filosofia pitagorica e propugnava l'intimo collegamento tra musica e matematica. In un certo senso, la perfetta architettura cui risponde l'opera si può perciò considerare un sia pur criptico proclama di filosofia pitagorica; il che verrebbe a confermare le ipotesi formulate da Hans-Eberhard Dentler nel 2000.